Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 1(3): 165-177
DOI:10.3934/Math.2016.3.165
Received: 22 July 2016
Accepted: 3 August 2016
Published: 15 August 2016

Research article

Applications of the Hille–Yosida theorem to the linearized equations of coupled sound and heat flow

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Abstract: This paper deals with the initial-value problem for the linearized equations of coupled sound and heat flow, in a bounded domain Ω in $\mathbb{R}^N$, with homogeneous Dirichlet boundary conditions. Existence and uniqueness of solutions to the problem are established by using the Hille–Yosida theorem. This paper gives a simpler proof than one by Carasso (1975). Moreover, regularity of solutions is established.

Keywords: coupled sound and heat flow; monotone operators; the Hille–Yosida theorem; existence; uniqueness; regularity of solutions

1. Introduction and results

We consider the following initial-boundary value problem:

$$\begin{cases} w_{tt} = c^2\Delta w - c^2\Delta e + m^2 w, & x \in \Omega,\ t > 0, \\ e_t = \sigma\Delta e - (\gamma - 1)w_t, & x \in \Omega,\ t > 0, \\ e = w = 0, & x \in \Gamma,\ t \geq 0, \\ w(x,0) = w_0(x),\ w_t(x,0) = v_0(x),\ e(x,0) = e_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $c > 0$, $\sigma > 0$, $m \in \mathbb{R}$ and $\gamma > 1$ are constants. We assume that Ω is a fixed domain in $\mathbb{R}^N$ and that the boundary $\Gamma := \partial\Omega$ is bounded and smooth. This problem originates from the following linearized equations of coupled sound and heat flow (cf. [3, 4]):

$$\frac{\partial w}{\partial t} = c\nabla \cdot \mathbf{u}, \tag{1.2}$$

$$\frac{\partial \mathbf{u}}{\partial t} = c\nabla w - c\nabla e, \tag{1.3}$$

$$\frac{\partial e}{\partial t} = \sigma\Delta e - (\gamma - 1)c\nabla \cdot \mathbf{u}. \tag{1.4}$$

As stated in [4, Section 1.4], these three equations appear in the flow of a compressible fluid. In such flow there are often considerable differences of temperature from one point to another, and the transfer of energy by thermal conduction may have a significant effect on the motion. The parabolic equation of heat flow is then coupled to the hyperbolic equations of fluid dynamics and the two phenomena must be calculated concurrently. This effect occurs also for infinitesimal or acoustic vibrations and is responsible for absorption of ultrasonic waves. Taking the divergence of both sides in (1.3) and eliminating $\nabla \cdot \mathbf{u}$ from the resulting system, we obtain two equations for the unknown scalar fields $w(x,t)$ and $e(x,t)$, namely, (1.2), (1.3), (1.4) are reduced to the two equations in (1.1) with $m = 0$.

Carasso [2] constructed and analyzed a least-squares procedure for approximately solving the problem (1.1) with $m = 0$. As a consequence existence and uniqueness of solutions were established.

The purpose of this paper is to give a simple proof of existence and uniqueness of solutions to (1.1) of Klein–Gordon type with $m \in \mathbb{R}$ by applying the Hille–Yosida theorem.

The first main result reads as follows.

Theorem 1.1 (existence and uniqueness)**.** *Assume that $w_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $v_0 \in H_0^1(\Omega)$ and $e_0 \in H^2(\Omega) \cap H_0^1(\Omega)$. Then there exists a unique solution (w, e) of* (1.1) *satisfying*

$$w \in C([0,\infty); H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,\infty); H_0^1(\Omega)) \cap C^2([0,\infty); L^2(\Omega)), \tag{1.5}$$

$$e \in C([0,\infty); H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,\infty); H_0^1(\Omega)) \cap C^2([0,\infty); L^2(\Omega)). \tag{1.6}$$

Moreover, for some $\alpha > 0$, the following estimates hold:

$$\begin{aligned} &\|w(t)\|_{H^1(\Omega)}^2 + \frac{1}{c^2}\|v(t)\|_{L^2(\Omega)}^2 + \frac{1}{\gamma - 1}\|e(t)\|_{H^1(\Omega)}^2 \\ &\le e^{2\alpha t}\left(\|w_0\|_{H^1(\Omega)}^2 + \frac{1}{c^2}\|v_0\|_{L^2(\Omega)}^2 + \frac{1}{\gamma - 1}\|e_0\|_{H^1(\Omega)}^2\right) \quad \forall\, t \ge 0, \end{aligned} \tag{1.7}$$

$$\begin{aligned} &\|v(t)\|_{H^1(\Omega)}^2 + \frac{1}{c^2}\|c^2\Delta w(t) - c^2\Delta e(t) + m^2 w(t)\|_{L^2(\Omega)}^2 + \frac{1}{\gamma - 1}\|\sigma\Delta e(t) - (\gamma - 1)v(t)\|_{H^1(\Omega)}^2 \\ &\le e^{2\alpha t}\left(\|v_0\|_{H^1(\Omega)}^2 + \frac{1}{c^2}\|c^2\Delta w_0 - c^2\Delta e_0 + m^2 w_0\|_{L^2(\Omega)}^2 + \frac{1}{\gamma - 1}\|\sigma\Delta e_0 - (\gamma - 1)v_0\|_{H^1(\Omega)}^2\right) \quad \forall\, t \ge 0. \end{aligned} \tag{1.8}$$

The second main result reads as follows.

Theorem 1.2 (regularity)**.** *Assume that the initial data w_0, v_0, e_0 satisfy*

$$w_0 \in H^k(\Omega),\ v_0 \in H^k(\Omega),\ e_0 \in H^k(\Omega) \qquad \forall\, k \in \mathbb{N},$$

and the compatibility conditions

$$\begin{aligned} \Delta^j w_0 &= 0 \quad \text{on } \Gamma \quad \forall\, j \ge 0, \quad j \text{ integer}, \\ \Delta^j v_0 &= 0 \quad \text{on } \Gamma \quad \forall\, j \ge 0, \quad j \text{ integer}, \\ \Delta^j e_0 &= 0 \quad \text{on } \Gamma \quad \forall\, j \ge 0, \quad j \text{ integer}. \end{aligned}$$

Then the solution (w, e) of (1.1) *belongs to $C^\infty(\overline{\Omega} \times [0,\infty)) \times C^\infty(\overline{\Omega} \times [0,\infty))$.*

This paper is organized as follows. In the following section we will rewrite the initial-boundary value problem (1.1) as the Cauchy problem for a single abstract evolution equation $dU/dt + AU = 0$, where A is a matrix of operators. We will also collect theorems in the Hille–Yosida theory which will be used in this paper. Section 3 is devoted to the proof of Theorem 1.1. In Section 4 we will give the proof of Theorem 1.2.

2. Abstract formulation toward the Hille–Yosida theory

Putting $v := w_t$, we rewrite equations in (1.1) as

$$\begin{cases} w_t = v, \\ v_t = c^2\Delta w - c^2\Delta e + m^2 w, \\ e_t = \sigma\Delta e - (\gamma-1)v, \end{cases} \tag{2.1}$$

so that (2.1) becomes

$$\begin{aligned} \begin{pmatrix} w_t \\ v_t \\ e_t \end{pmatrix} &= \begin{pmatrix} v \\ (c^2\Delta + m^2 I)w - c^2\Delta e \\ -(\gamma-1)v + \sigma\Delta e \end{pmatrix} \\ &= \begin{pmatrix} 0 & I & 0 \\ c^2\Delta + m^2 I & 0 & -c^2\Delta \\ 0 & -(\gamma-1)I & \sigma\Delta \end{pmatrix} \begin{pmatrix} w \\ v \\ e \end{pmatrix} \\ &= -\begin{pmatrix} 0 & -I & 0 \\ -c^2\Delta - m^2 I & 0 & c^2\Delta \\ 0 & (\gamma-1)I & -\sigma\Delta \end{pmatrix} \begin{pmatrix} w \\ v \\ e \end{pmatrix}. \end{aligned}$$

Setting

$$A := \begin{pmatrix} 0 & -I & 0 \\ -c^2\Delta - m^2 I & 0 & c^2\Delta \\ 0 & (\gamma-1)I & -\sigma\Delta \end{pmatrix}, \qquad U := \begin{pmatrix} w \\ v \\ e \end{pmatrix}, \tag{2.2}$$

we rewrite (1.1) as

$$\begin{cases} \dfrac{dU}{dt} + AU = 0, \quad t > 0, \\ U(0) = U_0, \end{cases} \tag{2.3}$$

where

$$U_0 := \begin{pmatrix} w_0 \\ v_0 \\ e_0 \end{pmatrix}.$$

We also note that

$$AU = \begin{pmatrix} -v \\ -c^2\Delta w + c^2\Delta e - m^2 w \\ -\sigma\Delta e + (\gamma-1)v \end{pmatrix}. \tag{2.4}$$

The following definition plays an important role in the Hille–Yosida theory.

Definition 2.1. An unbounded linear operator $A : D(A) \subset H \to H$ is said to be *monotone* if it satisfies

$$(Av, v) \geq 0 \quad \forall\, v \in D(A).$$

It is said to be *maximal monotone* if, in addition, $R(I + A) = H$, i.e.,

$$\forall\, f \in H\ \exists\, u \in D(A) \text{ such that } u + Au = f.$$

We next introduce two useful theorems (for the proof see [1, Chapter 7]).

Theorem 2.1 (Hille–Yosida)**.** *Let A be a maximal monotone operator in a Hilbert space H. Then, given any $u_0 \in D(A)$ there exists a unique function*

$$u \in C^1([0,\infty); H) \cap C([0,\infty); D(A))$$

satisfying

$$\begin{cases} \dfrac{du}{dt} + Au = 0 \quad \text{on } [0,\infty), \\ u(0) = u_0. \end{cases} \tag{2.5}$$

Moreover,

$$\|u(t)\| \leq \|u_0\| \quad \textit{and} \quad \left\|\frac{du}{dt}(t)\right\| = \|Au(t)\| \leq \|Au_0\| \quad \forall\, t \geq 0.$$

Theorem 2.2. *Assume $u_0 \in D(A^k)$ for some integer $k \geq 2$. Then the solution u of* (2.5) *obtained in Theorem* 2.1 *satisfies*

$$u \in C^{k-j}([0,\infty); D(A^j)) \quad \forall\, j = 0, 1, \ldots, k.$$

In order to apply the Hille–Yosida theory to (2.3) derived from (1.1) we define the domain of A given by (2.2) as

$$D(A) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \times (H^2(\Omega) \cap H_0^1(\Omega)).$$

Then A is an operator in the Hilbert space

$$H := H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega)$$

equipped with inner product

$$(U_1, U_2) := \int_\Omega \nabla w_1 \nabla w_2 + \int_\Omega w_1 w_2 + \frac{1}{c^2}\int_\Omega v_1 v_2 + \frac{1}{\gamma - 1}\int_\Omega \nabla e_1 \nabla e_2 + \frac{1}{\gamma - 1}\int_\Omega e_1 e_2,$$

where

$$U_j := \begin{pmatrix} w_j \\ v_j \\ e_j \end{pmatrix} \qquad (j = 1, 2).$$

Also, the norm in H is given by

$$\|U\|^2 = (U, U) = \|w\|^2_{H^1(\Omega)} + \frac{1}{c^2}\|v\|^2_{L^2(\Omega)} + \frac{1}{\gamma - 1}\|e\|^2_{H^1(\Omega)} \quad \text{for } U = \begin{pmatrix} w \\ v \\ e \end{pmatrix}. \tag{2.6}$$

In particular, we see from (2.4) that

$$\|AU\|^2 = \|v\|^2_{H^1(\Omega)} + \frac{1}{c^2}\|c^2\Delta w - c^2\Delta e + m^2 w\|^2_{L^2(\Omega)} + \frac{1}{\gamma - 1}\|\sigma\Delta e - (\gamma - 1)v\|^2_{H^1(\Omega)}. \tag{2.7}$$

3. Existence and uniqueness

In this section we prove Theorem 1.1 by using Theorem 2.1.

3.1. Monotonicity

Let A and H be as in the end of Section 2. Let

$$U = \begin{pmatrix} w \\ v \\ e \end{pmatrix} \in D(A).$$

Then it follows from the definition of the inner product and integration by part that

$$\begin{aligned}(AU, U) = &\int_\Omega \nabla(-v)\nabla w - \int_\Omega vw + \frac{1}{c^2}\int_\Omega v(-c^2\Delta w - m^2 w + c^2\Delta e)\\ &+ \frac{1}{\gamma-1}\int_\Omega \nabla[(\gamma-1)v - \sigma\Delta e]\nabla e + \frac{1}{\gamma-1}\int_\Omega e\,[(\gamma-1)v - \sigma\Delta e]\\ = &-\left(1+\frac{m^2}{c^2}\right)\int_\Omega vw + \frac{\sigma}{\gamma-1}\int_\Omega |\Delta e|^2 + \frac{\sigma}{\gamma-1}\int_\Omega |\nabla e|^2 + \int_\Omega ve.\end{aligned}$$

Since the second and third terms on the right-hand side are nonnegative, we have

$$\begin{aligned}(AU, U) &\geq -\left(1+\frac{m^2}{c^2}\right)\int_\Omega vw + \int_\Omega ve\\ &\geq -\left(1+\frac{m^2}{c^2}\right)\int_\Omega |v||w| - \int_\Omega |v||e|\\ &\geq -\frac{1+\frac{m^2}{c^2}}{2}\int_\Omega (v^2+w^2) - \frac{1}{2}\int_\Omega (v^2+e^2)\\ &= -\left(1+\frac{m^2}{2c^2}\right)\int_\Omega v^2 - \left(\frac{1}{2}+\frac{m^2}{2c^2}\right)\int_\Omega w^2 - \frac{1}{2}\int_\Omega e^2.\end{aligned}$$

We define a positive constant α_0 as

$$\alpha_0 := \max\left\{\frac{1}{2}+\frac{m^2}{2c^2},\quad c^2\left(1+\frac{m^2}{2c^2}\right),\quad \frac{\gamma-1}{2}\right\}.$$

Then we conclude that $A+\alpha_0$ is monotone:

$$\begin{aligned}((A+\alpha_0)U, U) \geq &-\left(1+\frac{m^2}{2c^2}\right)\int_\Omega v^2 - \left(\frac{1}{2}+\frac{m^2}{2c^2}\right)\int_\Omega w^2 - \frac{1}{2}\int_\Omega e^2\\ &+ \alpha_0\int_\Omega w^2 + \frac{\alpha_0}{c^2}\int_\Omega v^2 + \frac{\alpha_0}{\gamma-1}\int_\Omega e^2\\ &+ \alpha_0\int_\Omega |\nabla w|^2 + \frac{\alpha_0}{\gamma-1}\int_\Omega |\nabla e|^2\end{aligned}$$

$$
\begin{aligned}
&= \left[\alpha_0 - \left(\frac{1}{2} + \frac{m^2}{2c^2}\right)\right]\int_\Omega w^2 + \left[\frac{\alpha_0}{c^2} - \left(1 + \frac{m^2}{2c^2}\right)\right]\int_\Omega v^2 \\
&\quad + \left(\frac{\alpha_0}{\gamma - 1} - \frac{1}{2}\right)\int_\Omega e^2 + \alpha_0\int_\Omega |\nabla w|^2 + \frac{\alpha_0}{\gamma - 1}\int_\Omega |\nabla e|^2 \\
&\geq 0.
\end{aligned}
$$

3.2. Maximal monotonicity

We divide the proof into four steps.

Step 1. Writing down the aim. We will select $\alpha > 0$ later. The aim is to show that $A + \alpha I$ is maximal monotone, i.e.,

$$\forall\ F \in H \quad \exists\ U \in D(A) \quad \text{s.t.} \quad U + (A + \alpha I)U = F.$$

To see this we take

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix} \in H.$$

Then we shall find

$$
\begin{pmatrix} w \\ v \\ e \end{pmatrix} \in D(A) \quad \text{s.t.} \quad
\begin{cases}
-v + (\alpha + 1)w & = f, \\
-c^2\Delta w - m^2 w + c^2\Delta e + (\alpha + 1)v & = g, \\
(\gamma - 1)v - \sigma\Delta e + (\alpha + 1)e & = h.
\end{cases}
$$

Step 2. Reducing the equations. We first delete v and then we have

$$-c^2\Delta w + \left[(\alpha + 1)^2 - m^2\right] w + c^2\Delta e = (\alpha + 1)f + g, \tag{3.1}$$

$$-\sigma\Delta e + (\alpha + 1)e + (\alpha + 1)(\gamma - 1)w = (\gamma - 1)f + h. \tag{3.2}$$

Now let $\delta \neq 0$ which will be fixed later. Making (3.1) $\times\ \delta$ + (3.2), we have

$$
\begin{aligned}
-\delta c^2\Delta w - (\sigma - \delta c^2)\Delta e + \left[\delta(\alpha + 1)^2 - \delta m^2 + (\alpha + 1)(\gamma - 1)\right] w + (\alpha + 1)e& \\
= [\delta(\alpha + 1) + (\gamma - 1)]f + \delta g + h.&
\end{aligned} \tag{3.3}
$$

If there exists a constant k such that

$$\frac{\sigma - c^2\delta}{\delta c^2} = \frac{\alpha + 1}{\delta(\alpha + 1)^2 - \delta m^2 + (\alpha + 1)(\gamma - 1)} = k, \tag{3.4}$$

then (3.3) is reduced to

$$-\delta c^2\Delta u + [\delta(\alpha + 1)^2 - \delta m^2 + (\alpha + 1)(\gamma - 1)]u = [\delta(\alpha + 1) + (\gamma - 1)]f + \delta g + h, \tag{3.5}$$

where $u := w + ke$.

Step 3. Finding two kinds of (δ, k) *in* (3.4). We rewrite (3.4) as

$$\delta c^2(\alpha+1) = (\sigma - \delta c^2)[\delta(\alpha+1)^2 - \delta m^2 + (\alpha+1)(\gamma-1)].$$

Dividing the both sides, we have

$$\begin{aligned}\delta c^2 &= (\sigma - \delta c^2)\left[\delta(\alpha+1) - \frac{\delta m^2}{\alpha+1} + (\gamma-1)\right]\\ &= \sigma(\alpha+1)\delta - \frac{\sigma m^2}{\alpha+1}\delta + \sigma(\gamma-1) - c^2(\gamma-1) - c^2(\alpha+1)\delta^2 + \frac{c^2m^2}{\alpha+1}\delta^2 - c^2(\gamma-1)\delta.\end{aligned}$$

Therefore,

$$\varphi(\delta) := c^2\left[(\alpha+1) - \frac{m^2}{\alpha+1}\right]\delta^2 + \left[c^2\gamma - \sigma(\alpha+1) + \frac{\sigma m^2}{\alpha+1}\right]\delta - \gamma(\gamma-1) = 0. \tag{3.6}$$

In order to find two solutions $\delta = \delta_1, \delta_2$ of this equation we show that the discriminant D is positive for $\alpha > |m| - 1$. Indeed, we observe that

$$D = \left[c^2\gamma - \sigma(\alpha+1) + \frac{\sigma m^2}{\alpha+1}\right]^2 + 4c^2\left[(\alpha+1) - \frac{m^2}{\alpha+1}\right]\gamma(\gamma-1),$$

where if we take α as

$$\alpha > |m| - 1, \quad \text{i.e.,} \quad \alpha + 1 > |m|,$$

then

$$(\alpha+1) - \frac{m^2}{\alpha+1} = \frac{(\alpha+1)^2 - m^2}{\alpha+1} > 0.$$

Thus we deduce that $D > 0$. Noting that $\varphi(0) = -\gamma(\gamma-1) < 0$, we see that (3.6) has two solutions $\delta = \delta_1$, δ_2 such that $\delta_1 < 0$ and $\delta_2 > 0$:

$$\begin{aligned}\delta_1 &= \frac{-\left[c^2\gamma - \sigma(\alpha+1) + \frac{\sigma m^2}{\alpha+1}\right] - \sqrt{D}}{2c^2\left[(\alpha+1) - \frac{m^2}{\alpha+1}\right]},\\ \delta_2 &= \frac{-\left[c^2\gamma - \sigma(\alpha+1) + \frac{\sigma m^2}{\alpha+1}\right] + \sqrt{D}}{2c^2\left[(\alpha+1) - \frac{m^2}{\alpha+1}\right]}.\end{aligned}$$

So we find two kinds of (δ, k) in (3.4):

$$\begin{aligned}\frac{\sigma - c^2\delta}{\delta c^2} &= \frac{\alpha+1}{\delta(\alpha+1)^2 - \delta m^2 + (\alpha+1)(\gamma-1)}\\ &= \begin{cases} k_1, & \text{if} \quad \delta = \delta_1,\\ k_2, & \text{if} \quad \delta = \delta_2.\end{cases}\end{aligned}$$

Step 4. Conclusion. Let us consider the 2 parameters of Step 3, $\delta = \delta_1(< 0)$ and $\delta = \delta_2(> 0)$. Note that $k_1 \neq k_2$. Hence we can find two solutions $u = u_1, u_2$ of (3.5) with $\delta = \delta_1, \delta_2$, respectively (see [1, Theorems 9.21 and 9.25]):

$$\begin{aligned}
&-\delta_1 c^2 \Delta u_1 + [\delta_1(\alpha+1)^2 - \delta_1 m^2 + (\alpha+1)(\gamma-1)]u_1 \\
&\qquad\qquad = [\delta_1(\alpha+1) + (\gamma-1)]f + \delta_1 g + h, \qquad (3.7)\\
&-\delta_2 c^2 \Delta u_2 + [\delta_2(\alpha+1)^2 - \delta_2 m^2 + (\alpha+1)(\gamma-1)]u_2 \\
&\qquad\qquad = [\delta_2(\alpha+1) + (\gamma-1)]f + \delta_2 g + h, \qquad (3.8)
\end{aligned}$$

which are equivalent to

$$-c^2\Delta u_1 + \left[(\alpha+1)^2 - m^2 + \frac{(\alpha+1)(\gamma-1)}{\delta_1}\right]u_1 = \left[(\alpha+1) + \frac{(\gamma-1)}{\delta_1}\right]f + g + \frac{h}{\delta_1}, \qquad (3.9)$$

$$-c^2\Delta u_2 + \left[(\alpha+1)^2 - m^2 + \frac{(\alpha+1)(\gamma-1)}{\delta_2}\right]u_2 = \left[(\alpha+1) + \frac{(\gamma-1)}{\delta_2}\right]f + g + \frac{h}{\delta_2}, \qquad (3.10)$$

where the coefficients of the second terms on the left-hand sides are positive for some $\alpha > 0$. Indeed, the coefficient of u_2 is positive when $\alpha > |m| - 1$, because $\delta_2 > 0$. As to the coefficient of u_1, we see that

$$\begin{aligned}
\frac{1}{\delta_1} &= -\frac{2c^2\left[(\alpha+1) - \frac{m^2}{\alpha+1}\right]}{\left[c^2\gamma - \sigma(\alpha+1) + \frac{\sigma m^2}{\alpha+1}\right] + \sqrt{D}} \\
&= -\frac{2c^2\left[(\alpha+1) - \frac{m^2}{\alpha+1}\right]\left\{\left[c^2\gamma - \sigma(\alpha+1) + \frac{\sigma m^2}{\alpha+1}\right] - \sqrt{D}\right\}}{-4c^2\left[(\alpha+1) - \frac{m^2}{\alpha+1}\right]\gamma(\gamma-1)} \\
&= \frac{\left[c^2\gamma - \sigma(\alpha+1) + \frac{\sigma m^2}{\alpha+1}\right] - \sqrt{D}}{2\gamma(\gamma-1)}.
\end{aligned}$$

Hence it follows that

$$\begin{aligned}
&(\alpha+1)^2 - m^2 + \frac{(\alpha+1)(\gamma-1)}{\delta_1} \\
&= (\alpha+1)^2 - m^2 + (\alpha+1)\frac{\left[c^2\gamma - \sigma(\alpha+1) + \frac{\sigma m^2}{\alpha+1}\right] - \sqrt{D}}{2\gamma} > 0 \\
\Longleftrightarrow\ &(2\gamma - \sigma)(\alpha+1)^2 + c^2\gamma(\alpha+1) + (\sigma - 2\gamma)m^2 > (\alpha+1)\sqrt{D}. \qquad (3.11)
\end{aligned}$$

Let $\alpha > |m| - 1$. First consider the case $2\gamma \geq \sigma$. In this case, since

$$\begin{aligned}
&(2\gamma - \sigma)(\alpha+1)^2 + c^2\gamma(\alpha+1) + (\sigma - 2\gamma)m^2 \\
&= (2\gamma - \sigma)\left[(\alpha+1)^2 - m^2\right] + c^2\gamma(\alpha+1) \\
&\geq 0,
\end{aligned}$$

we have

$$
\begin{aligned}
& (2\gamma-\sigma)(\alpha+1)^2+c^2\gamma(\alpha+1)+(\sigma-2\gamma)m^2>(\alpha+1)\sqrt{D} \\
\Longleftrightarrow\quad & \left[(2\gamma-\sigma)(\alpha+1)^2+c^2\gamma(\alpha+1)+(\sigma-2\gamma)m^2\right]^2>(\alpha+1)^2 D \\
\Longleftrightarrow\quad & (\gamma-\sigma)(\alpha+1)^4+c^2(\alpha+1)^3-2(\gamma-\sigma)m^2(\alpha+1)^2 \\
& \qquad\qquad\qquad -c^2m^2(\alpha+1)+(\gamma-\sigma)m^4>0 \\
\Longleftrightarrow\quad & (\gamma-\sigma)\left[(\alpha+1)^2-m^2\right]^2+c^2(\alpha+1)\left[(\alpha+1)^2-m^2\right]>0 \qquad (3.12) \\
\Longleftrightarrow\quad & \left[(\alpha+1)^2-m^2\right]\left[(\gamma-\sigma)(\alpha+1)^2+c^2(\alpha+1)-(\gamma-\sigma)m^2\right]>0. \qquad (3.13)
\end{aligned}
$$

Therefore, if $\gamma \geq \sigma$, then the coefficient of u_1 in (3.9) is positive in view of (3.12). If $2\gamma \geq \sigma > \gamma$, then from (3.13) it suffices to choose α such that

$$(\gamma-\sigma)(\alpha+1)^2+c^2(\alpha+1)-(\gamma-\sigma)m^2>0,$$

that is,

$$(\sigma-\gamma)(\alpha+1)^2-c^2(\alpha+1)-(\sigma-\gamma)m^2<0.$$

Solving this inequality and noting that $\alpha+1>|m|\geq 0$, we have

$$(|m|<)\ \alpha+1<\frac{c^2+\sqrt{c^4+4(\sigma-\gamma)^2m^2}}{2(\sigma-\gamma)}. \tag{3.14}$$

Next consider the case $2\gamma<\sigma$. In this case, from (3.11) it suffices to take α such that

$$(2\gamma-\sigma)(\alpha+1)^2+c^2\gamma(\alpha+1)+(\sigma-2\gamma)m^2>0,$$

that is,

$$(\sigma-2\gamma)(\alpha+1)^2-c^2\gamma(\alpha+1)-(\sigma-2\gamma)m^2<0.$$

Solving this inequality gives

$$(|m|<)\ \alpha+1<\frac{c^2\gamma+\sqrt{c^4\gamma^2+4(\sigma-2\gamma)^2m^2}}{2(\sigma-2\gamma)}. \tag{3.15}$$

Hence the same way as in the case $2\gamma \geq \sigma$ yields that the coefficient of u_1 in (3.9) is positive when α satisfies (3.14) and (3.15). Thus we can find two solutions $u = u_1, u_2$ of (3.5) with $\delta = \delta_1, \delta_2$, respectively. For k_1, k_2 and u_1, u_2 constructed above, we solve the following system with respect to w, e:

$$\begin{cases} w+k_1e=u_1, \\ w+k_2e=u_2. \end{cases}$$

Then we find

$$w = \frac{1}{k_2 - k_1}(k_2 u_1 - k_1 u_2), \tag{3.16}$$

$$e = \frac{1}{k_2 - k_1}(u_2 - u_1). \tag{3.17}$$

Moreover, setting

$$v = \frac{\alpha + 1}{k_2 - k_1}(k_2 u_1 - k_1 u_2) - f, \tag{3.18}$$

we shall show that w, v, e are the desired functions in Step 1. Indeed, we see that (3.18) implies the required equation

$$-v + (\alpha + 1)w = f. \tag{3.19}$$

Making $(k_2 - k_1)[(3.16) + k_1 \times (3.17)]$ and $(k_2 - k_1)[(3.16) + k_2 \times (3.17)]$, we have

$$\begin{aligned}
&(k_1 - k_2)(w + k_1 e) = (k_1 - k_2)u_1, \text{ i.e., } u_1 = w + k_1 e,\\
&(k_1 - k_2)(w + k_2 e) = (k_1 - k_2)u_2, \text{ i.e., } u_2 w + k_2 e.
\end{aligned}$$

Therefore, in view of (3.7) and (3.8),

$$\begin{aligned}
&-\delta_1 c^2 \Delta(w + k_1 e) + [\delta_1(\alpha + 1)^2 - \delta_1 m^2 + (\alpha + 1)(\gamma - 1)](w + k_1 e) = [\delta_1(\alpha + 1) + (\gamma - 1)]f + \delta_1 g + h,\\
&-\delta_2 c^2 \Delta(w + k_2 e) + [\delta_2(\alpha + 1)^2 - \delta_2 m^2 + (\alpha + 1)(\gamma - 1)](w + k_2 e) = [\delta_2(\alpha + 1) + (\gamma - 1)]f + \delta_2 g + h,
\end{aligned}$$

of which the first equation is equivalent to

$$\begin{aligned}
&-\delta_1 c^2 \Delta w + [\delta_1(\alpha + 1)^2 - \delta_1 m^2 + (\alpha + 1)(\gamma - 1)]w\\
&\quad + [-\delta_1 c^2 \Delta e + \delta_1(\alpha + 1)^2 e - \delta_1 m^2 e + (\alpha + 1)(\gamma - 1)e]k_1\\
&\qquad = [\delta_1(\alpha + 1) + (\gamma - 1)]f + \delta_1 g + h.
\end{aligned}$$

Recall the definition of k_1:

$$\frac{\sigma - c^2\delta_1}{\delta_1 c^2} = \frac{\alpha + 1}{\delta_1(\alpha + 1)^2 - \delta_1 m^2 + (\alpha + 1)(\gamma - 1)} = k_1.$$

Then it follows that

$$k_1\delta_1 c^2 = \sigma - c^2\delta_1, \quad k_1\left[\delta_1(\alpha + 1)^2 - \delta_1 m^2 + (\alpha + 1)(\gamma - 1)\right] = \alpha + 1,$$

and hence

$$\begin{aligned}
&-\delta_1 c^2 \Delta w - (\sigma - \delta_1 c^2)\Delta e + \left[\delta_1(\alpha + 1)^2 - \delta_1 m^2 + (\alpha + 1)(\gamma - 1)\right] w + (\alpha + 1)e\\
&\qquad = [\delta_1(\alpha + 1) + (\gamma - 1)]f + \delta_1 g + h.
\end{aligned} \tag{3.20}$$

In the same way as above we can deduce

$$-\delta_2 c^2\Delta w-(\sigma-\delta_2 c^2)\Delta e+\Big[\delta_2(\alpha+1)^2-\delta_2 m^2+(\alpha+1)(\gamma-1)\Big]w+(\alpha+1)e$$
$$=[\delta_2(\alpha+1)+(\gamma-1)]f+\delta_2 g+h. \tag{3.21}$$

Making (3.20) − (3.21) and (3.20) × δ_2 − (3.21) × δ_1, we have

$$-(\delta_1-\delta_2)c^2\Delta w+(\delta_1-\delta_2)c^2\Delta e+(\delta_1-\delta_2)(\alpha+1)^2 w-(\delta_1-\delta_2)m^2 w$$
$$=(\delta_1-\delta_2)(\alpha+1)f+(\delta_1-\delta_2)g,$$

$$-(\delta_2-\delta_1)\sigma\Delta e(\delta_2-\delta_1)(\alpha+1)(\gamma-1)w+(\delta_2-\delta_1)(\alpha+1)e$$
$$=(\delta_2-\delta_1)(\gamma-1)f+(\delta_2-\delta_1)h.$$

Thus we arrive at (3.1) and (3.2). Making (3.1) − (3.19) × $(\alpha+1)$ and (3.2) − (3.19) × $(\gamma-1)$, we obtain

$$-c^2\Delta w-m^2 w+c^2\Delta e+(\alpha+1)v=g,$$

$$(\gamma-1)v-\sigma\Delta e+(\alpha+1)e=h.$$

Consequently, we conclude that w, v, e are the desired functions which satisfy

$$\begin{cases}-v+(\alpha+1)w & =f,\\ -c^2\Delta w-m^2 w+c^2\Delta e+(\alpha+1)v & =g,\\ (\gamma-1)v-\sigma\Delta e+(\alpha+1)e & =h.\end{cases}$$

3.3. Proof of Theorem 1.1

Since $A+\alpha$ is maximal monotone as proved above, it follows from Theorem 2.1 that for $U_0\in D(A)$ the problem

$$\begin{cases}\dfrac{dV}{dt}+AV+\alpha V=0 & \text{on } [0,\infty),\\ V(0)=U_0\end{cases}$$

has a unique solution $V\in C^1([0,\infty);H)\cap C([0,\infty);D(A))$ such that

$$\|V(t)\|\le\|U_0\| \quad \text{and} \quad \|AV(t)\|\le\|AU_0\| \quad \forall\, t\ge 0.$$

Setting

$$U(t):=e^{\alpha t}V(t),$$

we deduce that $U\in C^1([0,\infty);H)\cap C([0,\infty);D(A))$ satisfies

$$\begin{cases}\dfrac{dU}{dt}+AU=0 & \text{on } [0,\infty),\\ U(0)=U_0,\end{cases}$$

with the estimates

$$\|U(t)\|\le e^{\alpha t}\|U_0\| \quad \text{and} \quad \|AU(t)\|\le e^{\alpha t}\|AU_0\| \quad \forall\, t\ge 0.$$

The properties (1.5), (1.6), (1.7), (1.8) follow from those for U. This completes the proof of Theorem 1.1. □

4. Regularity

We use the same notation as in the end of Section 2.

Proof of Theorem 1.2. We first recall the definition of $D(A^k)$ which is given by induction as follows:

$$D(A^1) := D(A), \quad D(A^k) := \{U \in D(A^{k-1}) | \, AU \in D(A^{k-1})\}, \quad k \geq 2.$$

It is easy to see, by induction on k, that

$$D(A^k) = D_k := \left\{ \begin{pmatrix} w \\ v \\ e \end{pmatrix} \middle| \begin{array}{lll} w \in H^{k+1}(\Omega), & \Delta^j w = 0 \text{ on } \Gamma & (0 \leq \forall\, j \leq [\frac{k}{2}]) \\ v \in H^k(\Omega), & \Delta^j v = 0 \text{ on } \Gamma & (0 \leq \forall\, j \leq [\frac{k+1}{2}] - 1) \\ e \in H^{k+1}(\Omega), & \Delta^j e = 0 \text{ on } \Gamma & (0 \leq \forall\, j \leq [\frac{k}{2}]) \end{array} \right\}.$$

Indeed, when $k = 1$, we have

$$\begin{aligned} D(A^1) &= (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \times (H^2(\Omega) \cap H_0^1(\Omega)) \\ &= \left\{ \begin{pmatrix} w \\ v \\ e \end{pmatrix} \middle| \begin{array}{lll} w \in H^2(\Omega), & w = 0 & \text{on } \Gamma \\ v \in H^1(\Omega), & v = 0 & \text{on } \Gamma \\ e \in H^2(\Omega), & e = 0 & \text{on } \Gamma \end{array} \right\} = D_1. \end{aligned}$$

If $D(A^k) = D_k$ holds for k, then the statement for $k+1$ reads as follows:

$$\begin{aligned} D(A^{k+1}) &= \{U \in D(A^k) = D_k | \, AU \in D(A^k) = D_k\} \\ &= \left\{ \begin{pmatrix} w \\ v \\ e \end{pmatrix} \middle| \begin{array}{llll} w \in H^{k+1}(\Omega), & \Delta^j w = 0 & \text{on } \Gamma & (0 \leq \forall j \leq [\frac{k}{2}]) \\ v \in H^k(\Omega), & \Delta^j v = 0 & \text{on } \Gamma & (0 \leq \forall j \leq [\frac{k+1}{2}] - 1) \\ e \in H^{k+1}(\Omega), & \Delta^j e = 0 & \text{on } \Gamma & (0 \leq \forall j \leq [\frac{k}{2}]) \end{array} \right\} \\ &\quad \cap \left\{ \begin{pmatrix} w \\ v \\ e \end{pmatrix} \middle| \begin{array}{l} -v \in H^{k+1}(\Omega), \ \Delta^j(-v) = 0 \text{ on } \Gamma \ (0 \leq \forall j \leq [\frac{k}{2}]) \\ -c^2 \Delta w - m^2 w + c^2 \Delta e \in H^k(\Omega), \\ \Delta^j(-c^2 \Delta w - m^2 w + c^2 \Delta e) = 0 \text{ on } \Gamma \ (0 \leq \forall j \leq [\frac{k+1}{2}] - 1) \\ (\gamma - 1)v - \sigma \Delta e \in H^{k+1}(\Omega), \\ \Delta^j((\gamma - 1)v - \sigma \Delta e) = 0 \text{ on } \Gamma \ (0 \leq \forall j \leq [\frac{k}{2}]) \end{array} \right\} \\ &= D_{k+1}. \end{aligned}$$

In particular, $D(A^k) \subset H^{k+1}(\Omega) \times H^k(\Omega) \times H^{k+1}(\Omega)$ with continuous injection. Applying Theorem 2.2, we see that if $U_0 \in D(A^k)$, then the solution U of (2.3) satisfies

$$\begin{aligned} U &\in C^{k-j}([0, \infty]; D(A^j)) \\ &\subset C^{k-j}([0, \infty]; H^{j+1}(\Omega) \times H^j(\Omega) \times H^{j+1}(\Omega)) \quad \forall\, j = 0, 1, \ldots, k. \end{aligned}$$

Therefore we conclude by [1, Corollary 9.15] that under the assumption of Theorem 1.2 (i.e., $U_0 \in D(A^k) \ \forall\, k \in \mathbb{N}$), $U \in C^k([0, \infty); C^k(\Omega) \times C^k(\Omega) \times C^k(\Omega)) \ \forall\, k \in \mathbb{N}$. □

Acknowledgments

The authors would like to the anonymous referees for helpful suggestions. T. Yokota is supported by Grant-in-Aid for Scientific Research (C), No. 16K05182.

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